

# THE MINIMAL SURFACES OVER THE SLANTED HALF-PLANES, VERTICAL STRIPS AND SINGLE SLIT

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**ABSTRACT.** In this paper, we discuss the minimal surfaces over the slanted half-planes, vertical strips, and single slit whose slit lies on the negative real axis. The representation of these minimal surfaces and the corresponding harmonic mappings are obtained explicitly. Finally, we illustrate the harmonic mappings of each of these cases together with their minimal surfaces pictorially with the help of mathematica.

## 1. INTRODUCTION

A planar harmonic mapping in the unit disk  $\mathbb{D} = \{z : |z| < 1\}$  is a complex-valued harmonic function  $f(z)$ , defined on  $\mathbb{D}$ . The mapping  $f$  has a canonical decomposition  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic on  $\mathbb{D}$  and  $g(0) = 0$ . The mapping  $f$  is locally univalent in  $\mathbb{D}$  if and only if its Jacobian  $J_f(z) = |h'(z)|^2 - |g'(z)|^2$  does not vanish in  $\mathbb{D}$ . It is said to be sense-preserving on  $\mathbb{D}$  if and only if  $J_f(z) > 0$ , or equivalently if  $h'(z) \neq 0$  in  $\mathbb{D}$  and  $f$  satisfies the elliptic partial differential equation

$$\overline{f_{\bar{z}}(z)} = \omega(z)f_z(z)$$

in  $\mathbb{D}$ , where the dilatation  $\omega(z) = g'(z)/h'(z)$  has the property that  $|\omega(z)| < 1$  in  $\mathbb{D}$ .

Planar univalent harmonic mappings are used in the study of the Gaussian curvature of nonparametric minimal surfaces over simply connected domains (see for example [4, 5]). After the publication of landmark paper of Clunie and Sheil-Small [1], considerable interest in the function theoretic properties of harmonic functions, quite apart from this connection, was generated. Since then the study of univalent harmonic mappings has gained much attention. The case where  $\omega(z)$  is a finite Blaschke product is of special interest since this case arises in many different contexts (see [7, 12]). In the present paper we shall explicitly study the connection between certain classes of harmonic univalent mappings and the theory of minimal surfaces.

Let  $S$  be a nonparametric minimal surface over a simply connected domain  $\Omega$  in  $\mathbb{C}$  given by

$$S = \{(u, v, F(u, v)) : u + iv \in \Omega\},$$

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2000 *Mathematics Subject Classification.* Primary: 30C65, 30C45; Secondary: 30C20.

*Key words and phrases.* Univalent Harmonic mapping, slanted half-plane mapping, slit mapping, strip mapping, convex in the real direction, minimal surface.

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File: LiSamy3.tex, printed: 2-3-2013, 16.27.

The research of the first author was supported by NSF of Hunan (No. 10JJ4005), Hunan Provincial Education Department (No. 11B019) and partly supported by the construct program of the key discipline in Hunan province.

where we have identified  $\mathbb{R}^2$  with the complex plane in describing the domain of  $F$ . The following result due to Weierstrass-Enneper representation provides the close link between harmonic univalent mappings and the associated minimal surfaces. Then  $S$  is a minimal surface if and only if  $S$  has the representation of the form

$$S = \left\{ \left( \operatorname{Re} \int_0^z \phi_1(t) dt + c_1, \operatorname{Re} \int_0^z \phi_2(t) dt + c_2, \operatorname{Re} \int_0^z \phi_3(t) dt + c_3 \right) : z \in \mathbb{D} \right\},$$

where  $\phi_1, \phi_2, \phi_3$  are analytic in  $\mathbb{D}$ ,

$$(1) \quad \phi_1^2 + \phi_2^2 + \phi_3^2 = 0, \text{ and } f = u + iv = \operatorname{Re} \int_0^z \phi_1(t) dt + i \operatorname{Re} \int_0^z \phi_2(t) dt + c$$

is a sense-preserving univalent harmonic mapping from  $\mathbb{D}$  onto  $\Omega$ . For this case, we call  $S$  a minimal surface over  $\Omega$  with the projection  $f = u + iv$ .

Further basic information about harmonic mappings and their relation to minimal surfaces may be found in the book of Duren [4]. For instance, the following formulation is well-known (see for instance [4, Section 10.2]).

**Theorem A.** *If  $f = h + \bar{g}$  is a harmonic mapping of the form (1) with the dilatation  $\omega = b^2$ , where  $b(z) = \pm z$ , then we have*

$$\phi_1 = h' + g', \quad \phi_2 = -i(h' - g'), \quad \phi_3 = 2ibh'.$$

Using this, Jun [8] has considered the minimal surfaces associated with the harmonic mappings especially when  $\Omega = \{w : \operatorname{Im} w > 0\}$ . His main result, which is easy to prove, will now be recalled for the sake of convenient reference.

**Theorem B.** ([8]) *Let  $\Omega = \{w : \operatorname{Im} w > 0\}$  and  $p = p_1 + ip_2$  be a fixed point in  $\Omega$ , where  $p_1, p_2 \in \mathbb{R}$ . If  $S$  is a minimal surface over  $\Omega$  with the projection  $f = h + \bar{g}$ , where  $\omega(z) = \frac{g'(z)}{h'(z)} = b^2(z) = z^2$ ,  $b(z) = \pm z$  and  $f(0) = p$ , then  $S = \{(u, v, F(u, v)) : u + iv \in \Omega\}$ , where*

$$\begin{aligned} u &= \operatorname{Re} f(z) = p_1 + \frac{ip_2}{2} \left[ \left( \frac{1}{2} \log \frac{1+z}{1-z} + \frac{z}{(1-z)^2} \right) - \overline{\left( \frac{1}{2} \log \frac{1+z}{1-z} + \frac{z}{(1-z)^2} \right)} \right], \\ v &= \operatorname{Im} f(z) = \frac{p_2}{2} \left[ \frac{1+z}{1-z} + \overline{\left( \frac{1+z}{1-z} \right)} \right], \\ F &= \pm p_2 \operatorname{Re} \left( \frac{z}{(1-z)^2} - \frac{1}{2} \log \frac{1+z}{1-z} \right). \end{aligned}$$

The class  $\mathcal{S}_H$  of sense-preserving harmonic univalent mappings  $f = h + \bar{g}$  (normalized so that  $f(0) = 0 = h(0)$  and  $f_z(0) = 1$ ) together with its many geometric subclasses have been extensively studied (see [1, 4]). Let  $\mathcal{S}_H^0$  be the subset of all  $f \in \mathcal{S}_H$  in which  $b_1 = f_{\bar{z}}(0) = 0$ . We remark that the familiar class  $\mathcal{S}$  of normalized analytic univalent functions is contained in  $\mathcal{S}_H^0$ . Every  $f \in \mathcal{S}_H$  admits the complex dilatation  $\omega$  of  $f$  which satisfies  $|\omega(z)| < 1$  in  $\mathbb{D}$ . When  $f \in \mathcal{S}_H^0$ , we also have  $\omega'(0) = 0$ .

In this paper, we discuss the minimal surfaces over the slanted half-planes, vertical strips, and single slit whose slit lies on the negative real axis. Slanted half-plane mappings

are well suited in the study of convolution of harmonic mappings (see [3]). Since the slanted half-planes and vertical strips are convex domains, the following result of Clunie and Sheil-Small is applicable for these cases.

**Lemma C.** [1] *If  $f = h + \bar{g}$  is a sense-preserving univalent mapping such that  $f(\mathbb{D})$  is a convex domain, then the function  $h + e^{i\beta}g$  is univalent for each  $\beta$ ,  $0 \leq \beta < 2\pi$ .*

## 2. SLANTED HALF-PLANE MAPPINGS

Throughout this section, we let  $H_\gamma := \{w : \operatorname{Re}(e^{i\gamma}w) > -1/2\}$  be a slanted half-plane with the parameter  $\gamma$ , where  $0 \leq \gamma < 2\pi$ .

**Theorem 1.** *Let  $S$  be a minimal surface over  $H_\gamma$  with the projection  $f = h + \bar{g}$ , whose dilatation  $\omega = g'/h' = b^2$ , where  $b(z) = \pm z$ . Then*

$$S = \{(u, v, F(u, v)) : u + iv \in H_\gamma\} = \{(u(z), v(z), F(u(z), v(z))) : z \in \mathbb{D}\},$$

where

$$\begin{aligned} u &= \frac{\pi \sin \gamma}{4} - \cos \gamma - \operatorname{Im} \left( \frac{\sin \gamma}{4} \log \frac{z - e^{-i\gamma}}{z + e^{-i\gamma}} + \frac{\sin 2\gamma}{4(z - e^{-i\gamma})} \right) \\ &\quad - \operatorname{Re} \left( \frac{\cos \gamma}{2(z - e^{-i\gamma})^2} + \frac{3}{4(z - e^{-i\gamma})} \right), \\ v &= \frac{\pi \cos \gamma}{4} + \sin \gamma - \operatorname{Im} \left( \frac{\cos \gamma}{4} \log \frac{z - e^{-i\gamma}}{z + e^{-i\gamma}} - \frac{3}{4(z - e^{-i\gamma})} \right) \\ &\quad + \operatorname{Re} \left( \frac{\sin 2\gamma}{4(z - e^{-i\gamma})} - \frac{\sin \gamma}{2(z - e^{-i\gamma})^2} \right), \\ F &= \pm \operatorname{Re} \left[ \frac{\sin 2\gamma}{4} \log \frac{z + e^{-i\gamma}}{z - e^{-i\gamma}} + \frac{1}{2} e^{i(\gamma + \frac{\pi}{2})} \frac{1}{z - e^{-i\gamma}} + \frac{i}{2} \frac{1}{(z - e^{-i\gamma})^2} \right] + c, \end{aligned}$$

if  $\gamma \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}$ ;

$$\begin{aligned} u &= \operatorname{Im} \left( \frac{\sin \gamma}{2(1 - \sin 2\gamma)} \log \frac{z + ie^{i\gamma}}{ie^{i\gamma}} + \frac{\sin \gamma}{2(1 + \sin 2\gamma)} \log \frac{z - ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\sin \gamma}{\cos^2 2\gamma} \log \frac{z - e^{-i\gamma}}{-e^{-i\gamma}} \right) \\ &\quad - \frac{\cos \gamma}{\cos 2\gamma} - \frac{\cos \gamma}{\cos 2\gamma} \operatorname{Re} \frac{e^{-i\gamma}}{z - e^{-i\gamma}}, \\ v &= \operatorname{Im} \left( \frac{\cos \gamma}{2(1 - \sin 2\gamma)} \log \frac{z + ie^{i\gamma}}{ie^{i\gamma}} + \frac{\cos \gamma}{2(1 + \sin 2\gamma)} \log \frac{z - ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\cos \gamma}{\cos^2 2\gamma} \log \frac{z - e^{-i\gamma}}{-e^{-i\gamma}} \right) \\ &\quad - \frac{\sin \gamma}{\cos 2\gamma} - \frac{\sin \gamma}{\cos 2\gamma} \operatorname{Re} \frac{e^{-i\gamma}}{z - e^{-i\gamma}}, \\ F &= \pm \operatorname{Re} \left[ \frac{\log(z + ie^{i\gamma})}{2(1 - \sin 2\gamma)} - \frac{\log(z - ie^{i\gamma})}{2(1 + \sin 2\gamma)} - \frac{\sin 2\gamma}{\cos^2 2\gamma} \log(z - e^{-i\gamma}) - \frac{ie^{-i\gamma}}{(z - e^{-i\gamma}) \cos 2\gamma} \right] + c, \end{aligned}$$

if  $\gamma \notin \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}$ .

*Proof.* Let  $f = h + \bar{g} \in \mathcal{S}_H^0$  and  $f(\mathbb{D}) = H_\gamma$ . Then, we have

$$\operatorname{Re}(e^{i\gamma}f(z)) = \operatorname{Re}[e^{i\gamma}(h(z) + e^{-2i\gamma}g(z))] > -\frac{1}{2}, \quad z \in \mathbb{D},$$

so that  $(h + e^{-2i\gamma}g)(\mathbb{D}) = H_\gamma$  and by Lemma C,  $h + e^{-2i\gamma}g$  is conformal (univalent) mapping from  $\mathbb{D}$  onto  $H_\gamma$ .

We now consider the function  $h + e^{-2i\gamma}g$ . We may conveniently normalize it in such a way that  $f(0) = h(0) = g(0) = 0$ . Then  $h(0) + e^{-2i\gamma}g(0) = 0$ . We further assume that

$$h(e^{-i\gamma}) + e^{-2i\gamma}g(e^{-i\gamma}) = \infty \quad \text{and} \quad h(e^{-i(\pi+\gamma)}) + e^{-2i\gamma}g(e^{-i(\pi+\gamma)}) = -\frac{1}{2}e^{-i\gamma}.$$

By the uniqueness of the Riemann mapping theorem, these observations led to the representation (see also [3, Lemma 1])

$$(2) \quad h(z) + e^{-2i\gamma}g(z) = \frac{z}{1 - e^{i\gamma}z}$$

from which we obtain

$$(3) \quad g(z) = -\frac{1}{z - e^{-i\gamma}} - e^{2i\gamma}h(z) - e^{i\gamma}$$

and

$$h'(z) + e^{-2i\gamma}g'(z) = \frac{1}{(1 - e^{i\gamma}z)^2}.$$

Solving this together with  $g'(z) = z^2h'(z)$  gives

$$h'(z) = \frac{1}{(z^2 + e^{2i\gamma})(z - e^{-i\gamma})^2} \quad \text{and} \quad g'(z) = \frac{z^2}{(z^2 + e^{2i\gamma})(z - e^{-i\gamma})^2}.$$

It is convenient to write  $h'(z)$  in the form

$$(4) \quad h'(z) = \frac{1}{(z - e^{i(\gamma+\pi/2)})(z - e^{i(\gamma-\pi/2)})(z - e^{-i\gamma})^2}.$$

In order to determine  $h(z)$  explicitly, we need to decompose it into partial fractions, and it is also clear that we need to deal with the cases where

$$\gamma \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\} \quad \text{and} \quad \gamma \notin \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}.$$

**Case 1:** Let  $\gamma = \frac{\pi}{4}$ .

In this case,  $h'(z)$  given by (4) takes the form

$$h'(z) = \frac{1}{(z + e^{-\frac{i\pi}{4}})(z - e^{-\frac{i\pi}{4}})^3}$$

so that  $h'(z)$  has a simple pole at  $z = -e^{-\frac{i\pi}{4}}$  and a pole of order 3 at  $z = e^{-\frac{i\pi}{4}}$ . We see that

$$h'(z) = \frac{i}{8}e^{\frac{i\pi}{4}} \left( \frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{1}{z + e^{-\frac{i\pi}{4}}} \right) - \frac{i}{4} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2} + \frac{1}{2}e^{\frac{i\pi}{4}} \frac{1}{(z - e^{-\frac{i\pi}{4}})^3},$$

Integration from 0 to  $z$  gives

$$(5) \quad h(z) = \left[ \frac{1}{8} e^{\frac{3i\pi}{4}} \log \frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{i}{4} \frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{1}{4} e^{\frac{i\pi}{4}} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2} \right] - \frac{1}{2} e^{-\frac{i\pi}{4}} + \frac{\pi}{8} e^{\frac{i\pi}{4}}.$$

Equation (3) for  $\gamma = \frac{\pi}{4}$  gives

$$g(z) = -\frac{1}{z - e^{-\frac{i\pi}{4}}} - ih(z) - e^{\frac{i\pi}{4}}$$

so that

$$h(z) + g(z) = -\frac{1}{z - e^{-\frac{i\pi}{4}}} + \sqrt{2} e^{-\frac{i\pi}{4}} h(z) - e^{\frac{i\pi}{4}}$$

and thus, substituting the expression for  $h(z)$  defined by (5) yields that

$$h(z) + g(z) = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{8} \log \frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} - \frac{3-i}{4} \frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{\sqrt{2}}{4} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2}$$

and similarly

$$h(z) - g(z) = \frac{i\sqrt{2}\pi}{8} + \frac{i\sqrt{2}}{2} - \frac{\sqrt{2}}{8} \log \frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{3+i}{4} \frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{i\sqrt{2}}{4} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2}.$$

As  $u = \operatorname{Re} f(z) = \operatorname{Re} (h(z) + g(z))$  and  $v = \operatorname{Im} f(z) = \operatorname{Im} (h(z) - g(z))$ , the last two equalities give

$$u = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} - \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{1}{4} \frac{1}{z - e^{-\frac{i\pi}{4}}} \right) - \operatorname{Re} \left( \frac{\sqrt{2}}{4} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2} + \frac{3}{4} \frac{1}{z - e^{-\frac{i\pi}{4}}} \right),$$

and

$$v = \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} - \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} - \frac{3}{4} \frac{1}{z - e^{-\frac{i\pi}{4}}} \right) + \operatorname{Re} \left( \frac{1}{4} \frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{\sqrt{2}}{4} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2} \right).$$

Finally, as  $b(z) = \pm z$ , Theorem A gives,

$$\begin{aligned} \phi_3(z) &= 2ibh'(z) = \pm 2i \frac{z}{(z + e^{-\frac{i\pi}{4}})(z - e^{-\frac{i\pi}{4}})^3} \\ &= \pm 2i \left[ \frac{i}{8} \frac{1}{z + e^{-\frac{i\pi}{4}}} - \frac{i}{8} \frac{1}{z - e^{-\frac{i\pi}{4}}} + \frac{1}{4} e^{\frac{i\pi}{4}} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2} + \frac{1}{2} \frac{1}{(z - e^{-\frac{i\pi}{4}})^3} \right]. \end{aligned}$$

and therefore,

$$\begin{aligned} F(z) &= \operatorname{Re} \int_0^z \phi_3(z) dz + c \\ &= \mp \operatorname{Re} \left[ \frac{1}{4} \log \frac{z + e^{-\frac{i\pi}{4}}}{z - e^{-\frac{i\pi}{4}}} + \frac{1}{2} e^{\frac{3i\pi}{4}} \frac{1}{z - e^{-\frac{i\pi}{4}}} + \frac{i}{2} \frac{1}{(z - e^{-\frac{i\pi}{4}})^2} \right] + c. \end{aligned}$$

**Case 2:** Let  $\gamma = \frac{3\pi}{4}$ .

In this case,  $h'(z)$  given by (4) takes the form

$$h'(z) = \frac{1}{(z - e^{\frac{i\pi}{4}})(z + e^{\frac{i\pi}{4}})^3}$$

and the partial fraction expansion gives

$$h'(z) = \frac{1}{8}e^{\frac{i\pi}{4}} \left( \frac{1}{z + e^{\frac{i\pi}{4}}} - \frac{1}{z - e^{\frac{i\pi}{4}}} \right) + \frac{i}{4} \frac{1}{(z + e^{\frac{i\pi}{4}})^2} - \frac{1}{2}e^{-\frac{i\pi}{4}} \frac{1}{(z + e^{\frac{i\pi}{4}})^3}.$$

Integration from 0 to  $z$  gives

$$(6) \quad h(z) = \frac{1}{8}e^{\frac{i\pi}{4}} \log \frac{z + e^{\frac{i\pi}{4}}}{z - e^{\frac{i\pi}{4}}} - \frac{i}{4} \frac{1}{z + e^{\frac{i\pi}{4}}} + \frac{1}{4}e^{-\frac{i\pi}{4}} \frac{1}{(z + e^{\frac{i\pi}{4}})^2} + \frac{1}{2}e^{\frac{i\pi}{4}} - \frac{i\pi}{8}e^{\frac{i\pi}{4}}.$$

Using (3) for  $\gamma = \frac{3\pi}{4}$ , we see that

$$h(z) + g(z) = -\frac{1}{z + e^{\frac{i\pi}{4}}} + \sqrt{2}e^{\frac{i\pi}{4}}h(z) + e^{-\frac{i\pi}{4}}$$

and

$$h(z) - g(z) = \frac{1}{z + e^{\frac{i\pi}{4}}} + \sqrt{2}e^{-\frac{i\pi}{4}}h(z) - e^{-\frac{i\pi}{4}}$$

which, by (6), simplify to

$$h(z) + g(z) = \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{8} \log \frac{z + e^{\frac{i\pi}{4}}}{z - e^{\frac{i\pi}{4}}} - \frac{3+i}{4} \frac{1}{z + e^{\frac{i\pi}{4}}} + \frac{\sqrt{2}}{4} \frac{1}{(z + e^{\frac{i\pi}{4}})^2}$$

and

$$h(z) - g(z) = -\frac{i\sqrt{2}\pi}{8} + \frac{i\sqrt{2}}{2} + \frac{\sqrt{2}}{8} \log \frac{z + e^{\frac{i\pi}{4}}}{z - e^{\frac{i\pi}{4}}} + \frac{3-i}{4} \frac{1}{z + e^{\frac{i\pi}{4}}} - \frac{i\sqrt{2}}{4} \frac{1}{(z + e^{\frac{i\pi}{4}})^2},$$

respectively. As  $u = \operatorname{Re} f(z) = \operatorname{Re} (h(z) + g(z))$  and  $v = \operatorname{Im} f(z) = \operatorname{Im} (h(z) - g(z))$ , it follows easily that

$$u = \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} - \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z + e^{\frac{i\pi}{4}}}{z - e^{\frac{i\pi}{4}}} - \frac{1}{4} \frac{1}{z + e^{\frac{i\pi}{4}}} \right) + \operatorname{Re} \left( \frac{\sqrt{2}}{4} \frac{1}{(z + e^{\frac{i\pi}{4}})^2} - \frac{3}{4} \frac{1}{z + e^{\frac{i\pi}{4}}} \right),$$

and

$$v = -\frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} + \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z + e^{\frac{i\pi}{4}}}{z - e^{\frac{i\pi}{4}}} + \frac{3}{4} \frac{1}{z + e^{\frac{i\pi}{4}}} \right) - \operatorname{Re} \left( \frac{\sqrt{2}}{4} \frac{1}{(z + e^{\frac{i\pi}{4}})^2} + \frac{1}{4} \frac{1}{z + e^{\frac{i\pi}{4}}} \right).$$

Moreover, in this case Theorem A implies that

$$\begin{aligned} \phi_3(z) &= 2ibh'(z) = \pm 2i \frac{z}{(z - e^{\frac{i\pi}{4}})(z + e^{\frac{i\pi}{4}})^3} \\ &= \pm 2i \left[ -\frac{i}{8} \frac{1}{z - e^{\frac{i\pi}{4}}} + \frac{i}{8} \frac{1}{z + e^{\frac{i\pi}{4}}} + \frac{1}{4}e^{\frac{3i\pi}{4}} \frac{1}{(z + e^{\frac{i\pi}{4}})^2} + \frac{1}{2} \frac{1}{(z + e^{\frac{i\pi}{4}})^3} \right]. \end{aligned}$$

Integration from 0 to  $z$  gives

$$F(z) = \mp \operatorname{Re} \left[ -\frac{1}{4} \log \frac{z - e^{\frac{i\pi}{4}}}{z + e^{\frac{i\pi}{4}}} - \frac{1}{2} e^{\frac{i\pi}{4}} \frac{1}{z + e^{\frac{i\pi}{4}}} + \frac{i}{2} \frac{1}{(z + e^{\frac{i\pi}{4}})^2} \right] + c.$$

**Case 3:** Let  $\gamma = \frac{5\pi}{4}$ .

In this case,  $h'(z)$  given by (4) takes the form

$$h'(z) = \frac{1}{(z - e^{-\frac{i\pi}{4}})(z + e^{-\frac{i\pi}{4}})^3}$$

and therefore,

$$h'(z) = \frac{1}{8} e^{-\frac{i\pi}{4}} \left( \frac{1}{z + e^{-\frac{i\pi}{4}}} - \frac{1}{z - e^{-\frac{i\pi}{4}}} \right) - \frac{1}{4} e^{\frac{i\pi}{2}} \frac{1}{(z + e^{-\frac{i\pi}{4}})^2} - \frac{1}{2} e^{\frac{i\pi}{4}} \frac{1}{(z + e^{-\frac{i\pi}{4}})^3}.$$

Integration from 0 to  $z$  gives

$$(7) \quad h(z) = \frac{1}{8} e^{-\frac{i\pi}{4}} \log \frac{z + e^{-\frac{i\pi}{4}}}{z - e^{-\frac{i\pi}{4}}} + \frac{i}{4} \frac{1}{z + e^{-\frac{i\pi}{4}}} + \frac{1}{4} e^{\frac{i\pi}{4}} \frac{1}{(z + e^{-\frac{i\pi}{4}})^2} - \frac{\pi}{8} e^{\frac{i\pi}{4}} + \frac{1}{2} e^{-\frac{i\pi}{4}}.$$

Using (3) for  $\gamma = \frac{5\pi}{4}$ , we see that

$$h(z) + g(z) = -\frac{1}{z + e^{-\frac{i\pi}{4}}} + \sqrt{2} e^{-\frac{i\pi}{4}} h(z) + e^{\frac{i\pi}{4}}$$

and

$$h(z) - g(z) = \frac{1}{z + e^{-\frac{i\pi}{4}}} + \sqrt{2} e^{\frac{i\pi}{4}} h(z) - e^{\frac{i\pi}{4}}.$$

As in the earlier two cases, a routine computation with the help of (7) shows that

$$u = \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z + e^{-\frac{i\pi}{4}}}{z - e^{-\frac{i\pi}{4}}} - \frac{1}{4} \frac{1}{z + e^{-\frac{i\pi}{4}}} \right) + \operatorname{Re} \left( \frac{\sqrt{2}}{4} \frac{1}{(z + e^{-\frac{i\pi}{4}})^2} - \frac{3}{4} \frac{1}{z + e^{-\frac{i\pi}{4}}} \right) - \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2},$$

and

$$v = \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z + e^{-\frac{i\pi}{4}}}{z - e^{-\frac{i\pi}{4}}} + \frac{3}{4} \frac{1}{z + e^{-\frac{i\pi}{4}}} \right) + \operatorname{Re} \left( \frac{\sqrt{2}}{4} \frac{1}{(z + e^{-\frac{i\pi}{4}})^2} + \frac{1}{4} \frac{1}{z + e^{-\frac{i\pi}{4}}} \right) - \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2},$$

where  $u = \operatorname{Re}(h(z) + g(z))$  and  $v = \operatorname{Im}(h(z) - g(z))$ .

In this case, according to Theorem A, we have

$$\begin{aligned} \phi_3(z) &= 2ibh'(z) = \pm 2i \frac{z}{(z - e^{-\frac{i\pi}{4}})(z + e^{-\frac{i\pi}{4}})^3} \\ &= \pm 2i \left[ \frac{i}{8} \frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{i}{8} \frac{1}{z + e^{-\frac{i\pi}{4}}} - \frac{1}{4} e^{\frac{i\pi}{4}} \frac{1}{(z + e^{-\frac{i\pi}{4}})^2} + \frac{1}{2} \frac{1}{(z + e^{-\frac{i\pi}{4}})^3} \right]. \end{aligned}$$

Integration from 0 to  $z$  gives

$$F(z) = \mp \operatorname{Re} \left[ \frac{1}{4} \log \frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{e^{-\frac{i\pi}{4}}}{2} \frac{1}{z + e^{-\frac{i\pi}{4}}} + \frac{i}{2} \frac{1}{(z + e^{-\frac{i\pi}{4}})^2} \right] + c.$$

**Case 4:** Let  $\gamma = \frac{7\pi}{4}$ .

In this case,  $h'(z)$  given by (4) takes the form

$$h'(z) = \frac{1}{(z + e^{\frac{i\pi}{4}})(z - e^{\frac{i\pi}{4}})^3}$$

and therefore,

$$h'(z) = -\frac{1}{8}e^{\frac{i\pi}{4}} \left( \frac{1}{z - e^{\frac{i\pi}{4}}} - \frac{1}{z + e^{\frac{i\pi}{4}}} \right) + \frac{1}{4}e^{\frac{i\pi}{2}} \frac{1}{(z - e^{\frac{i\pi}{4}})^2} + \frac{1}{2}e^{-\frac{i\pi}{4}} \frac{1}{(z - e^{\frac{i\pi}{4}})^3}.$$

Integration from 0 to  $z$  gives

$$(8) \quad h(z) = -\frac{1}{8}e^{\frac{i\pi}{4}} \log \frac{z - e^{\frac{i\pi}{4}}}{z + e^{\frac{i\pi}{4}}} - \frac{i}{4} \frac{1}{z - e^{\frac{i\pi}{4}}} - \frac{e^{-\frac{i\pi}{4}}}{4} \frac{1}{(z - e^{\frac{i\pi}{4}})^2} - \frac{\pi}{8}e^{-\frac{i\pi}{4}} - \frac{1}{2}e^{\frac{i\pi}{4}}.$$

Using (3) for  $\gamma = \frac{7\pi}{4}$ , we find that

$$h(z) + g(z) = -\frac{1}{z - e^{\frac{i\pi}{4}}} + \sqrt{2}e^{\frac{i\pi}{4}}h(z) - e^{-\frac{i\pi}{4}}$$

and

$$h(z) - g(z) = \frac{1}{z - e^{\frac{i\pi}{4}}} + \sqrt{2}e^{-\frac{i\pi}{4}}h(z) + e^{-\frac{i\pi}{4}},$$

where  $h$  is defined by (8). We thus obtain that

$$u = -\frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z - e^{\frac{i\pi}{4}}}{z + e^{\frac{i\pi}{4}}} + \frac{1}{4} \frac{1}{z - e^{\frac{i\pi}{4}}} \right) - \operatorname{Re} \left( \frac{\sqrt{2}}{4} \frac{1}{(z - e^{\frac{i\pi}{4}})^2} + \frac{3}{4} \frac{1}{z - e^{\frac{i\pi}{4}}} \right),$$

and

$$v = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} - \operatorname{Im} \left( \frac{\sqrt{2}}{8} \log \frac{z - e^{\frac{i\pi}{4}}}{z + e^{\frac{i\pi}{4}}} - \frac{3}{4} \frac{1}{z - e^{\frac{i\pi}{4}}} \right) + \operatorname{Re} \left( \frac{\sqrt{2}}{4} \frac{1}{(z - e^{\frac{i\pi}{4}})^2} - \frac{1}{4} \frac{1}{z - e^{\frac{i\pi}{4}}} \right),$$

where  $u = \operatorname{Re}(h(z) + g(z))$  and  $v = \operatorname{Im}(h(z) - g(z))$ .

In this case, by Theorem A, we find that

$$\begin{aligned} \phi_3(z) &= 2ibh'(z) = \pm 2i \frac{z}{(z + e^{\frac{i\pi}{4}})(z - e^{\frac{i\pi}{4}})^3} \\ &= \pm 2i \left[ -\frac{i}{8} \frac{1}{z + e^{\frac{i\pi}{4}}} + \frac{i}{8} \frac{1}{z - e^{\frac{i\pi}{4}}} + \frac{1}{4}e^{-\frac{i\pi}{4}} \frac{1}{(z - e^{\frac{i\pi}{4}})^2} + \frac{1}{2} \frac{1}{(z - e^{\frac{i\pi}{4}})^3} \right]. \end{aligned}$$

Integration from 0 to  $z$  gives

$$F(z) = \mp \operatorname{Re} \left[ -\frac{1}{4} \log \frac{z + e^{\frac{i\pi}{4}}}{z - e^{\frac{i\pi}{4}}} + \frac{1}{2}e^{\frac{i\pi}{4}} \frac{1}{z - e^{\frac{i\pi}{4}}} + \frac{i}{2} \frac{1}{(z - e^{\frac{i\pi}{4}})^2} \right] + c.$$

**Case 5:** Let  $\gamma \notin \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}$ .

In this case,  $h'(z)$  given by (4) has simple poles at  $ie^{i\gamma}$  and  $-ie^{i\gamma}$ , and a pole of order 2 at  $e^{-i\gamma}$ . Thus, we may rewrite  $h'(z)$  as

$$h'(z) = \frac{A}{z + ie^{i\gamma}} + \frac{B}{z - ie^{i\gamma}} + \frac{C}{z - e^{-i\gamma}} + \frac{D}{(z - e^{-i\gamma})^2}$$



where  $A, B, C$  and  $D$  can be easily computed using a standard procedure from residue calculus or otherwise. Indeed

$$A = \frac{e^{-i\gamma}}{4(1 - \sin 2\gamma)}, \quad B = \frac{e^{-i\gamma}}{4(1 + \sin 2\gamma)}, \quad C = -\frac{e^{-i\gamma}}{2 \cos^2 2\gamma}, \quad \text{and} \quad D = \frac{1}{2 \cos 2\gamma}.$$

We observe that  $A + B + C = 0$ . Integration from 0 to  $z$  leads to

$$(9) \quad h(z) = A \log \frac{z + ie^{i\gamma}}{ie^{i\gamma}} + B \log \frac{z - ie^{i\gamma}}{-ie^{i\gamma}} + C \log \frac{z - e^{-i\gamma}}{-e^{-i\gamma}} - \frac{D}{z - e^{-i\gamma}} - De^{i\gamma}.$$

Note that  $g$  defined by (3) gives

$$h(z) + g(z) = -\frac{1}{z - e^{-i\gamma}} - 2ie^{i\gamma} \sin \gamma h(z) - e^{i\gamma}$$

and

$$h(z) - g(z) = \frac{1}{z - e^{-i\gamma}} + 2e^{i\gamma} \cos \gamma h(z) + e^{i\gamma}$$

where  $h$  is defined by (9). By computation, we know that

$$\begin{aligned} u &= \operatorname{Im} \left( \frac{\sin \gamma}{2(1 - \sin 2\gamma)} \log \frac{z + ie^{i\gamma}}{ie^{i\gamma}} + \frac{\sin \gamma}{2(1 + \sin 2\gamma)} \log \frac{z - ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\sin \gamma}{\cos^2 2\gamma} \log \frac{z - e^{-i\gamma}}{-e^{-i\gamma}} \right) \\ &\quad - \frac{\cos \gamma}{\cos 2\gamma} \operatorname{Re} \left( \frac{z}{z - e^{-i\gamma}} \right), \\ v &= \operatorname{Im} \left( \frac{\cos \gamma}{2(1 - \sin 2\gamma)} \log \frac{z + ie^{i\gamma}}{ie^{i\gamma}} + \frac{\cos \gamma}{2(1 + \sin 2\gamma)} \log \frac{z - ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\cos \gamma}{\cos^2 2\gamma} \log \frac{z - e^{-i\gamma}}{-e^{-i\gamma}} \right) \\ &\quad - \frac{\sin \gamma}{\cos 2\gamma} \operatorname{Re} \left( \frac{z}{z - e^{-i\gamma}} \right). \end{aligned}$$

In the final case, by Theorem A, we find that

$$\begin{aligned} \phi_3(z) &= 2ibh'(z) = \pm \frac{2iz}{(z^2 + e^{2i\gamma})(z - e^{-i\gamma})^2} \\ &= \pm 2i \left[ -\frac{i}{4(1 - \sin 2\gamma)(z + ie^{i\gamma})} + \frac{i}{4(1 + \sin 2\gamma)(z - ie^{i\gamma})} \right. \\ &\quad \left. + \frac{i \sin 2\gamma}{2(z - e^{-i\gamma}) \cos^2 2\gamma} + \frac{e^{-i\gamma}}{2(z - e^{-i\gamma})^2 \cos 2\gamma} \right]. \end{aligned}$$

Integration from 0 to  $z$  gives

$$F = \pm \operatorname{Re} \left[ \frac{\log(z + ie^{i\gamma})}{2(1 - \sin 2\gamma)} - \frac{\log(z - ie^{i\gamma})}{2(1 + \sin 2\gamma)} - \frac{\sin 2\gamma}{\cos^2 2\gamma} \log(z - e^{-i\gamma}) - \frac{ie^{-i\gamma}}{(z - e^{-i\gamma}) \cos 2\gamma} \right] + c.$$

The proof is complete.  $\square$

## 3. VERTICAL STRIPS

Hengartner and Schober [6] investigated the family of functions from  $\mathcal{S}_H$  that map  $\mathbb{D}$  onto the horizontal strip domain  $\{w : |\operatorname{Im} w| < \pi/4\}$ . As an analogous result, Dorff [2] considered the family  $\mathcal{S}_H(\mathbb{D}, \Omega_\alpha)$  of functions from  $\mathcal{S}_H$  which map  $\mathbb{D}$  onto the asymmetric vertical strip domains

$$\Omega_\alpha = \left\{ w : \frac{\alpha - \pi}{2 \sin \alpha} < \operatorname{Re} w < \frac{\alpha}{2 \sin \alpha} \right\},$$

where  $\frac{\pi}{2} \leq \alpha < \pi$ . Set  $\mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha) = \mathcal{S}_H(\mathbb{D}, \Omega_\alpha) \cap \mathcal{S}_H^0$ . Note that  $\Omega_{\pi/2} = \{w : |\operatorname{Re} w| < \pi/4\}$  and so, the class discussed by Hengartner and Schober [6] follows by using a suitable rotation.

**Lemma 1.** *Each  $f = h + \bar{g} \in \mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha)$  has the form*

$$(10) \quad h(z) + g(z) = \psi(z), \quad \psi(z) = \frac{1}{2i \sin \alpha} \log \left( \frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right).$$

Moreover,

$$(11) \quad h'(z) = \frac{\psi'(z)}{1 + \omega(z)}, \quad g'(z) = \frac{\omega(z)\psi'(z)}{1 + \omega(z)} \quad \text{and} \quad \psi'(z) = \frac{1}{(1 + ze^{-i\alpha})(1 + ze^{i\alpha})}.$$

Here  $\omega(z) = g'(z)/h'(z)$  denotes the dilatation of  $f$ .

*Proof.* The representation (10) is well-known whereas (11) follows if we solve the pair of equations  $h'(z) + g'(z) = \psi'(z)$  and  $\omega(z)h'(z) - g'(z) = 0$ . The proof is complete.  $\square$

**Theorem 2.** *Let  $S$  be a minimal surface over  $\Omega_\alpha$  with the projection  $f = h + \bar{g} \in \mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha)$ , which satisfies (1) and whose dilatation  $\omega = b^2$ , where  $b(z) = \pm z$ . Then  $S = \{(u, v, F(u, v)) : u + iv \in \Omega_\alpha\}$ , where*

$$\begin{aligned} u &= \frac{1}{2 \sin \alpha} \operatorname{Im} \left[ \log \left( \frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right) \right], \\ v &= \begin{cases} \operatorname{Im} \left( \frac{z}{z^2 + 1} \right) & \text{if } \alpha = \frac{\pi}{2}, \\ \frac{1}{2 \cos \alpha} \operatorname{Im} \left[ \log \left( \frac{(1 + ze^{i\alpha})(1 + ze^{-i\alpha})}{z^2 + 1} \right) \right] & \text{if } \frac{\pi}{2} < \alpha < \pi \end{cases} \end{aligned}$$

and

$$F = \begin{cases} \pm \operatorname{Im} \left( \frac{1}{z^2 + 1} \right) + c & \text{if } \alpha = \frac{\pi}{2}, \\ \pm \operatorname{Re} \left[ \frac{1}{2 \cos \alpha} \log \left( \frac{z + i}{z - i} \right) - \frac{1}{\sin 2\alpha} \log \left( \frac{z + e^{i\alpha}}{z + e^{-i\alpha}} \right) \right] + c & \text{if } \frac{\pi}{2} < \alpha < \pi. \end{cases}$$

*Proof.* Let  $f = h + \bar{g} \in \mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha)$  with  $\omega(z) = z^2$ . Then by Lemma 1, we have

$$(12) \quad h'(z) = \begin{cases} \frac{1}{(z + i)^2(z - i)^2} & \text{if } \alpha = \frac{\pi}{2}, \\ \frac{1}{(z + i)(z - i)(z + e^{i\alpha})(z + e^{-i\alpha})} & \text{if } \frac{\pi}{2} < \alpha < \pi. \end{cases}$$

**Case (i):** Let  $\alpha = \frac{\pi}{2}$ . Then consider the partial fraction expression for  $h'(z)$ :

$$h'(z) = \frac{1}{4} \left[ \frac{i}{z+i} - \frac{i}{z-i} - \frac{1}{(z+i)^2} - \frac{1}{(z-i)^2} \right].$$

Integration from 0 to  $z$  gives

$$(13) \quad h(z) = \frac{1}{4} \left[ i \log(z+i) - i \log(z-i) + \frac{1}{z+i} + \frac{1}{z-i} + \pi \right].$$

Also, by (10), we obtain that

$$(14) \quad h(z) + g(z) = \frac{1}{2i} \log \left( \frac{i-z}{i+z} \right) = \frac{1}{2} [i \log(z+i) - i \log(z-i) + \pi]$$

so that, by (13) and (14)

$$h(z) - g(z) = 2h(z) - (h(z) + g(z)) = \frac{z}{z^2 + 1}.$$

As before, it follows that

$$u = \frac{1}{2} \operatorname{Im} \left( \log \left( \frac{i-z}{i+z} \right) \right), \quad v = \operatorname{Im} \left( \frac{z}{z^2 + 1} \right)$$

and  $\phi_3$  given by Theorem A takes the form

$$\phi_3(z) = \pm \frac{2iz}{(1-iz)^2(1+iz)^2} = \pm \frac{1}{2} \left( \frac{1}{(z-i)^2} - \frac{1}{(z+i)^2} \right).$$

We thus obtain  $F$  by integration:

$$F = \pm \operatorname{Im} \left( \frac{1}{z^2 + 1} \right) + c.$$

**Case (ii):** Let  $\frac{\pi}{2} < \alpha < \pi$ . The partial fraction expansion of  $h'(z)$  in (12) yields

$$h'(z) = -\frac{1}{4 \cos \alpha} \left( \frac{1}{z+i} + \frac{1}{z-i} \right) + \frac{1}{(e^{-i\alpha} - e^{3i\alpha})(z + e^{i\alpha})} + \frac{1}{(e^{i\alpha} - e^{-3i\alpha})(z + e^{-i\alpha})}.$$

Integration from 0 to  $z$  gives

$$h(z) = -\frac{1}{4 \cos \alpha} \log(z^2 + 1) + \frac{1}{e^{-i\alpha} - e^{3i\alpha}} \log(1 + ze^{-i\alpha}) + \frac{1}{e^{i\alpha} - e^{-3i\alpha}} \log(1 + ze^{i\alpha}),$$

which simplifies to

$$(15) \quad h(z) = -\frac{1}{4 \cos \alpha} \log(z^2 + 1) + \frac{ie^{-i\alpha}}{2 \sin 2\alpha} \log(1 + ze^{-i\alpha}) - \frac{ie^{i\alpha}}{2 \sin 2\alpha} \log(1 + ze^{i\alpha}).$$

By using (10), we obtain that

$$u = \operatorname{Re}(h(z) + g(z)) = \frac{1}{2 \sin \alpha} \operatorname{Im} \left( \log \left( \frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right) \right).$$

Writing  $h(z) - g(z) = 2h(z) - (h(z) + g(z))$  and using (10) and (15), we can easily find that

$$h(z) - g(z) = -\frac{1}{2 \cos \alpha} \log(z^2 + 1) + \frac{1}{2 \cos \alpha} \log(1 + ze^{-i\alpha}) + \frac{1}{2 \cos \alpha} \log(1 + ze^{i\alpha})$$

which gives

$$v = \operatorname{Im}(h(z) - g(z)) = \frac{1}{2 \cos \alpha} \operatorname{Im} \left( \log \frac{(1 + ze^{i\alpha})(1 + ze^{-i\alpha})}{z^2 + 1} \right).$$

In this case,  $\phi_3$  given by Theorem A takes the form

$$\begin{aligned} \phi_3(z) &= \pm \frac{2iz}{(z+i)(z-i)(z+e^{i\alpha})(z+e^{-i\alpha})} \\ &= \pm 2i \left[ \frac{i}{4 \cos \alpha} \left( \frac{1}{z+i} - \frac{1}{z-i} \right) + \frac{1}{2i \sin 2\alpha} \left( \frac{1}{z+e^{i\alpha}} - \frac{1}{z+e^{-i\alpha}} \right) \right]. \end{aligned}$$

Integration from 0 to  $z$  gives

$$F = \pm \operatorname{Re} \left[ \frac{1}{2 \cos \alpha} \log \left( \frac{z+i}{z-i} \right) - \frac{1}{\sin 2\alpha} \log \left( \frac{z+e^{i\alpha}}{z+e^{-i\alpha}} \right) \right] + c$$

and the proof is complete.  $\square$

#### 4. SINGLE SLIT

Finally, we consider single slit domain  $L$  whose slit lies on the negative real axis. Moreover, by the result of Livingston [9] (see also [10] and Dorff [2, Corollary 2]) it follows that if  $f = h + \bar{g} \in \mathcal{S}_H^0$  is a slit mapping whose slit lies on the negative real axis, then one has

$$(16) \quad h(z) - g(z) = \frac{z}{(1-z)^2}.$$

**Theorem 3.** *Let  $S$  be a minimal surface over  $L$  with the projection  $f = h + \bar{g} \in \mathcal{S}_H^0$ , which satisfies (16) and whose dilatation  $\omega = b^2$ , where  $b(z) = \pm z$ . Then  $S = \{(u, v, F(u, v)) : u + iv \in L\}$ , where*

$$u = \operatorname{Re} \left( \frac{2z^3 - 3z^2 + 3z}{3(1-z)^3} \right), \quad v = \operatorname{Im} \left( \frac{z}{(1-z)^2} \right),$$

and

$$F = \pm \operatorname{Im} \left( \frac{1}{(z-1)^2} + \frac{2}{3(z-1)^3} \right) + c.$$

*Proof.* By assumption,  $f = h + \bar{g} \in \mathcal{S}_H^0$  is a single slit mapping whose slit lies on the negative real axis with  $\omega(z) = z^2$ . Then (16) holds and therefore, we have

$$h'(z) - g'(z) = \frac{1+z}{(1-z)^3} \quad \text{and} \quad g'(z) = z^2 h'(z).$$

Solving these two equations, we obtain

$$h'(z) = \frac{1}{(1-z)^4}.$$

Integrating from 0 to  $z$  yields

$$h(z) = -\frac{1}{3} + \frac{1}{3(1-z)^3}$$

and so

$$g(z) = h(z) - \frac{z}{(1-z)^2} = -\frac{1}{3} + \frac{1}{3(1-z)^3} - \frac{z}{(1-z)^2},$$

which, by using the previous equation, gives

$$h(z) + g(z) = \frac{2z^3 - 3z^2 + 3z}{3(1-z)^3}.$$

The desired representations for  $u = \operatorname{Re}(h(z) + g(z))$  and  $v = \operatorname{Im}(h(z) - g(z))$  follow easily. Finally, since

$$\phi_3(z) = \pm 2izh'(z) = \pm \frac{2iz}{(1-z)^4} = \pm 2i \left( \frac{1}{(1-z)^4} - \frac{1}{(1-z)^3} \right),$$

integrating this from 0 to  $z$  yields

$$F = \pm \operatorname{Im} \left( \frac{1}{(z-1)^2} + \frac{2}{3(z-1)^3} \right) + c.$$

The proof is complete. □

## 5. ILLUSTRATION USING MATHEMATICA

The images of the disk  $|z| < r$  for  $r$  closer to 1 under  $f = h + \bar{g}$  for various cases of Theorem 1 and the corresponding minimal surfaces associated with  $f$  are illustrated in Figures 1-8. Similar illustrations for Theorem 2 (see Figures 9-12) and Theorem 3 (see Figure 13) are also provided. These figures are drawn using Mathematica (see for example [11]).

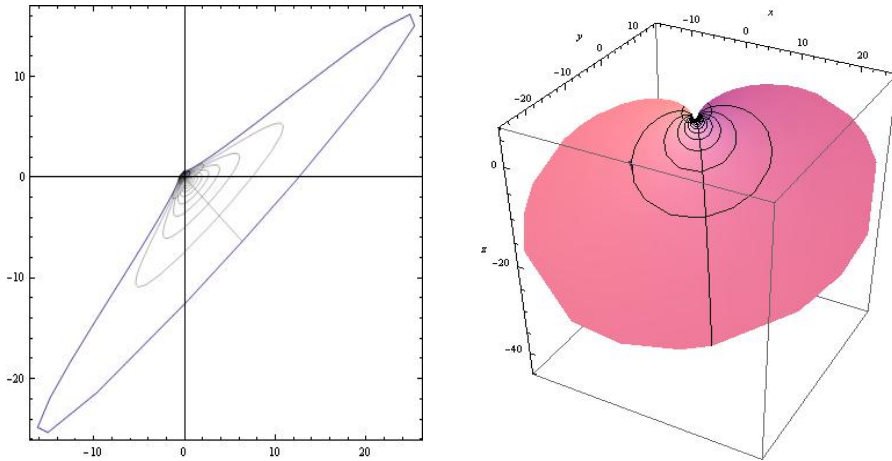
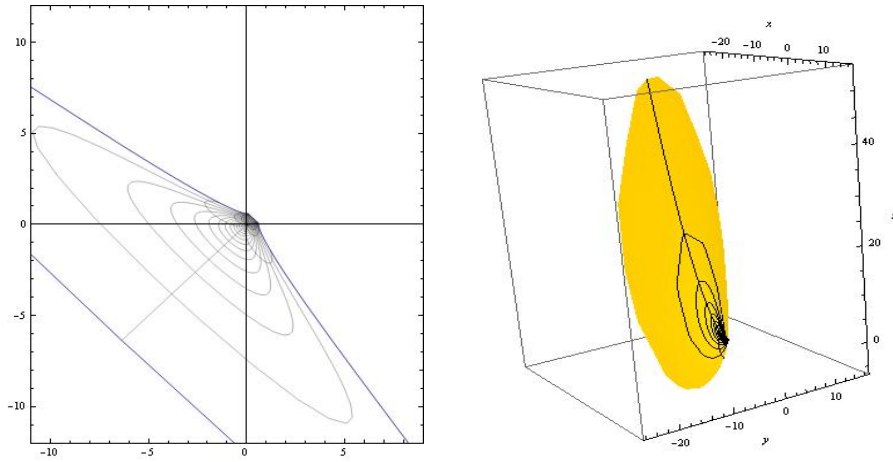
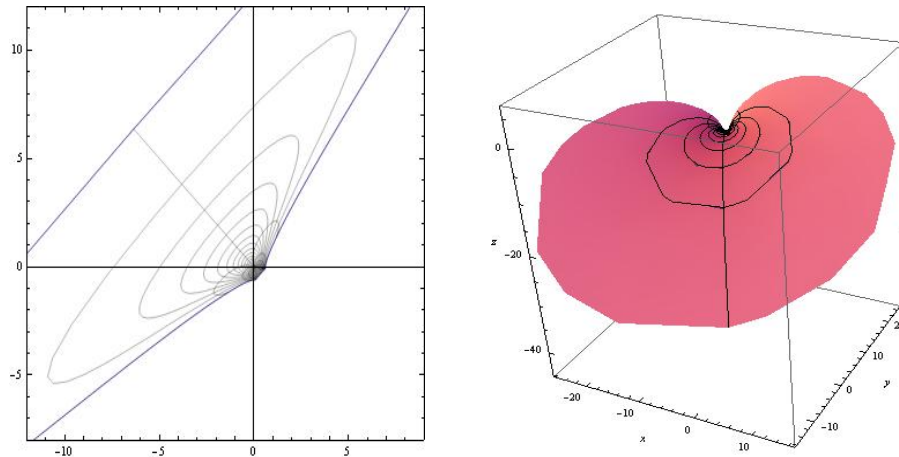
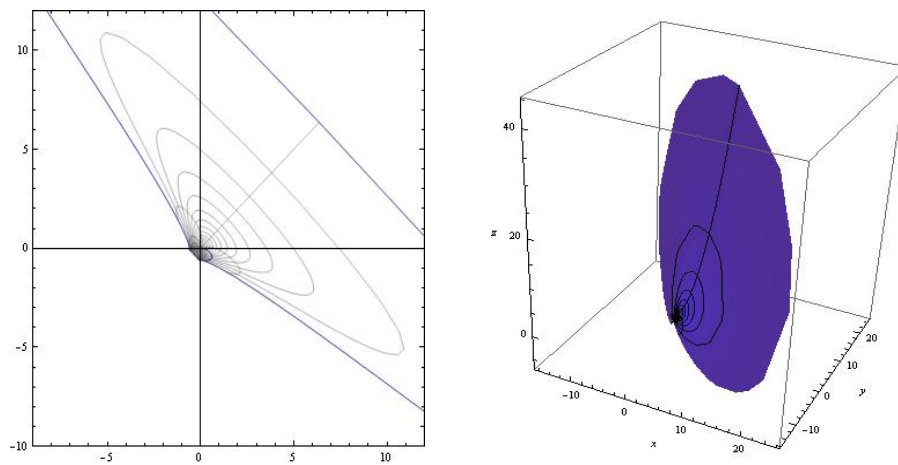


FIGURE 1. **Case 1:**  $\gamma = \pi/4$  of Theorem 1

FIGURE 2. **Case 2:**  $\gamma = 3\pi/4$  of Theorem 1FIGURE 3. **Case 3:**  $\gamma = 5\pi/4$  of Theorem 1FIGURE 4. **Case 4:**  $\gamma = 7\pi/4$  of Theorem 1

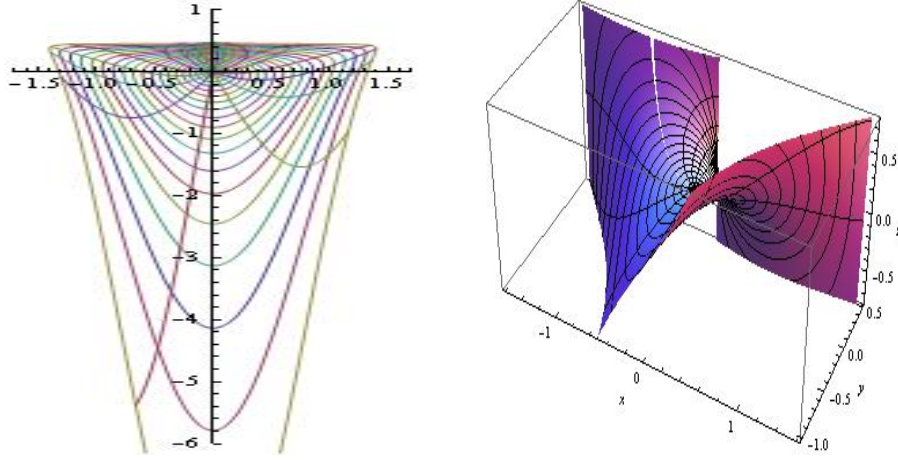


FIGURE 5. **Case 5** with  $\gamma = \pi/2$  of Theorem 1

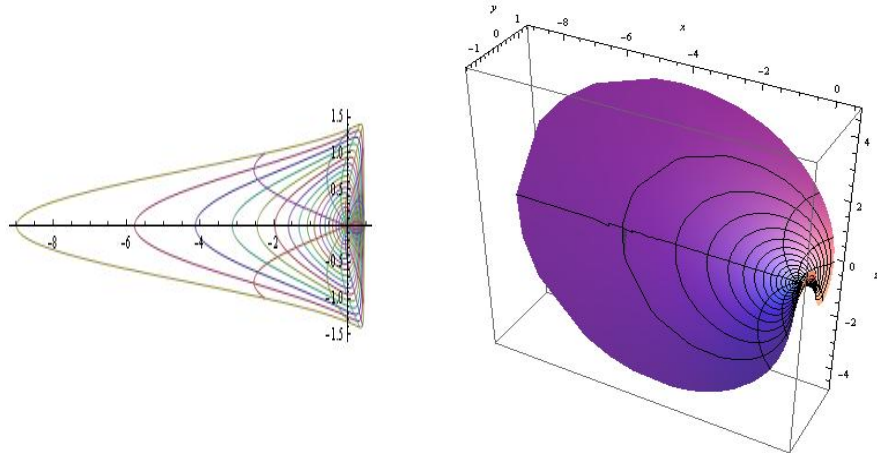


FIGURE 6. **Case 5** with  $\gamma = \pi$  of Theorem 1

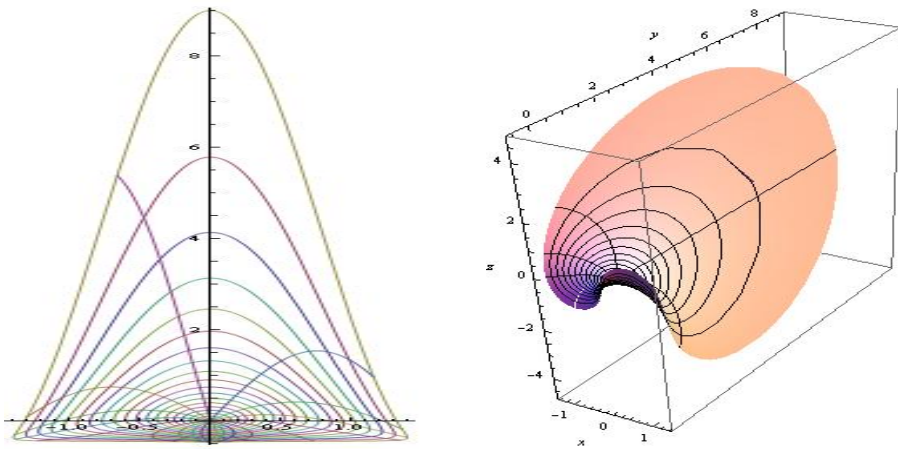
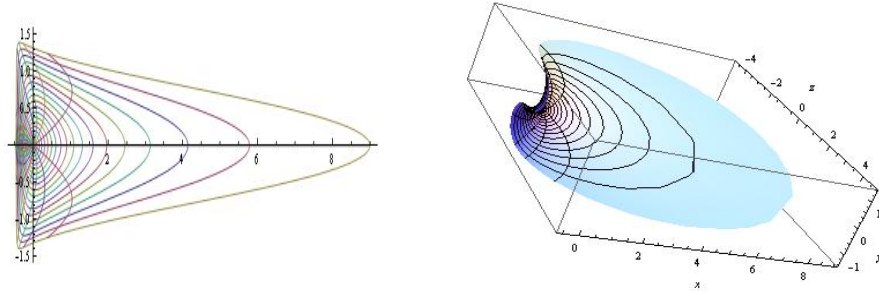
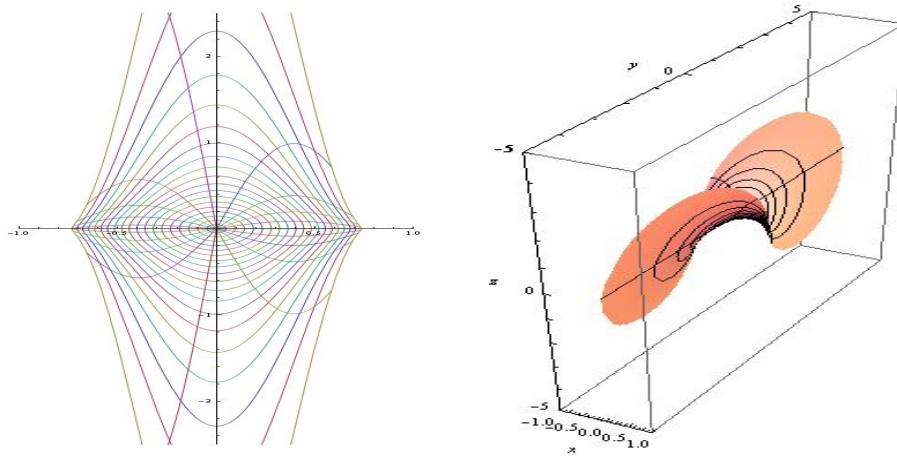
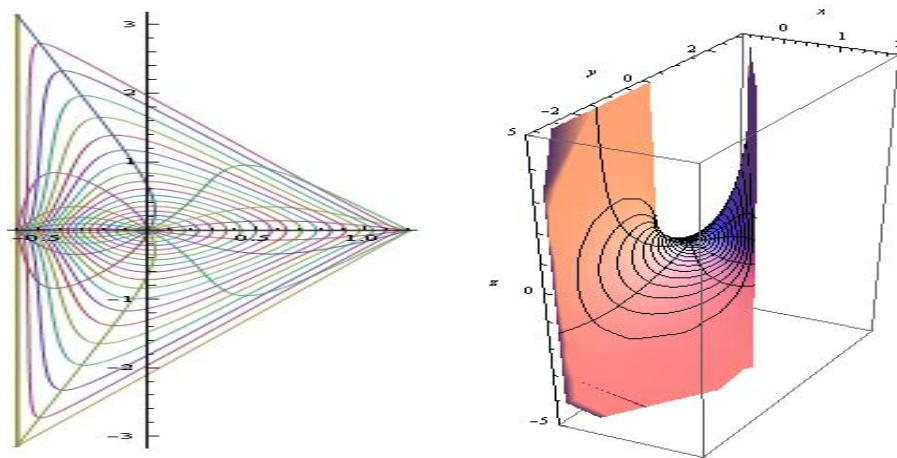


FIGURE 7. **Case 5** with  $\gamma = 3\pi/2$  of Theorem 1

FIGURE 8. **Case 5** with  $\gamma = 0$  of Theorem 1FIGURE 9. Illustration for  $\alpha = \pi/2$  of Theorem 2FIGURE 10. Illustration for  $\alpha = 2\pi/3$  of Theorem 2



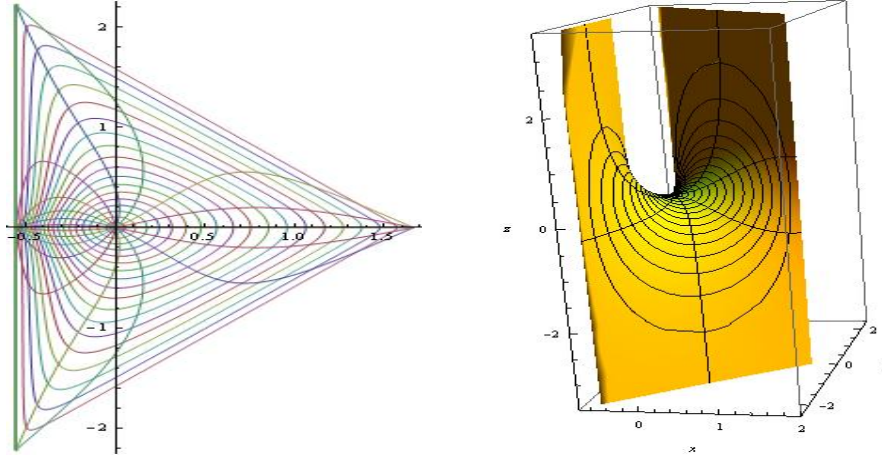


FIGURE 11. Illustration for  $\alpha = 3\pi/4$  of Theorem 2

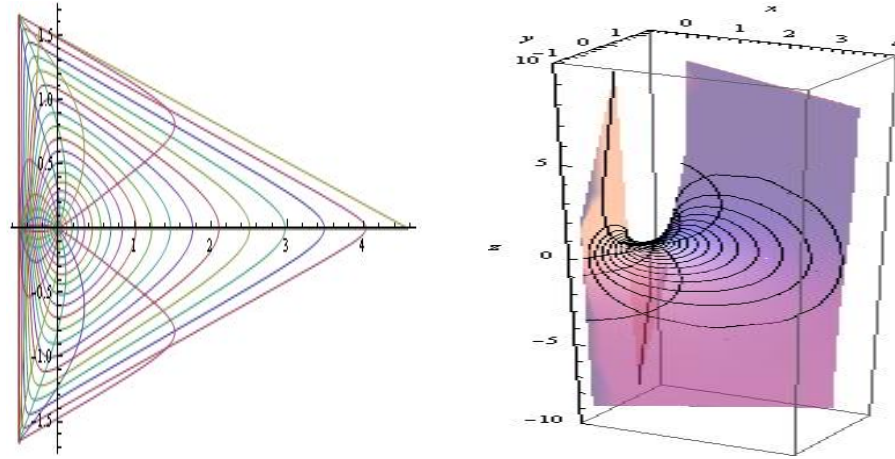


FIGURE 12. Illustration for  $\alpha = 9\pi/10$  of Theorem 2

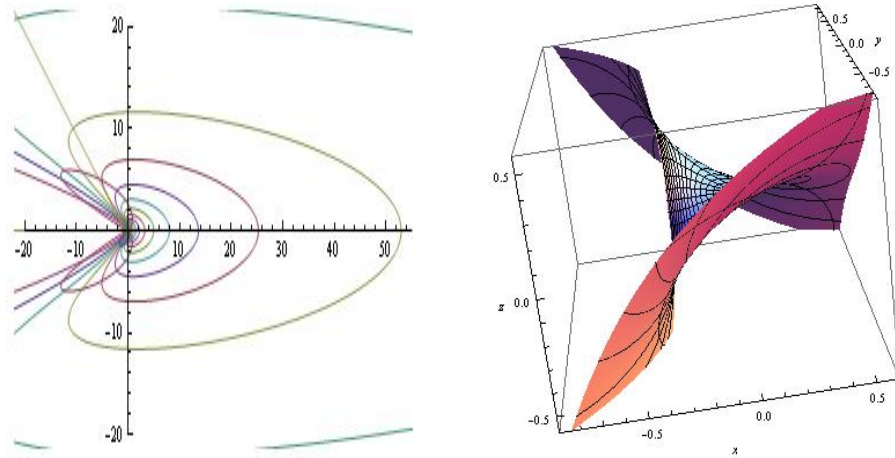


FIGURE 13. Illustration for Theorem 3

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